

# C4 DIFFERENTIATION

# Answers - Worksheet B

1 a  $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2t$

b  $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = 2t \div 1 = 2t$

2 a  $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3}{2t}$

b  $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 6t^2 + 2t$       c  $\frac{dx}{dt} = 2 \cos t, \frac{dy}{dt} = -6 \sin t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{6t^2 + 2t}{2t} = 3t + 1$        $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-6 \sin t}{2 \cos t} = -3 \tan t$

d  $\frac{dx}{dt} = 3, \frac{dy}{dt} = t^{-2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{t^{-2}}{3}$   
 $= \frac{1}{3t^2}$

e  $\frac{dx}{dt} = -2 \sin 2t, \frac{dy}{dt} = \cos t$       f  $\frac{dx}{dt} = e^{t+1}, \frac{dy}{dt} = 2e^{2t-1}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\cos t}{-2 \sin 2t}$        $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2e^{2t-1}}{e^{t+1}}$   
 $= \frac{\cos t}{-4 \sin t \cos t} = -\frac{1}{4} \operatorname{cosec} t$        $= 2e^{t-2}$

g  $\frac{dx}{dt} = 2 \sin t \times \cos t,$

$\frac{dy}{dt} = 3 \cos^2 t \times (-\sin t)$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-3 \cos^2 t \sin t}{2 \sin t \cos t}$   
 $= -\frac{3}{2} \cos t$

h  $\frac{dx}{dt} = 3 \sec t \tan t,$

$\frac{dy}{dt} = 5 \sec^2 t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{5 \sec^2 t}{3 \sec t \tan t}$   
 $= \frac{5 \sec t}{3 \tan t} = \frac{5}{3} \operatorname{cosec} t$

i  $\frac{dx}{dt} = -(t+1)^{-2},$

$\frac{dy}{dt} = \frac{1 \times (t-1) - t \times 1}{(t-1)^2} = -(t-1)^{-2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-(t-1)^{-2}}{-(t+1)^{-2}}$   
 $= \frac{(t+1)^2}{(t-1)^2} = \left(\frac{t+1}{t-1}\right)^2$

3 a  $t = 1 \therefore x = 1, y = 3$

$\frac{dx}{dt} = 3t^2, \frac{dy}{dt} = 6t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{6t}{3t^2} = \frac{2}{t}$

grad = 2

$\therefore y - 3 = 2(x - 1)$

$y = 2x + 1$

b  $t = 2 \therefore x = -3, y = 0$

$\frac{dx}{dt} = -2t, \frac{dy}{dt} = 2 - 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2-2t}{-2t} = 1 - \frac{1}{t}$

grad =  $\frac{1}{2}$

$\therefore y - 0 = \frac{1}{2}(x + 3)$

$y = \frac{1}{2}x + \frac{3}{2}$

c  $t = \frac{\pi}{3} \therefore x = \sqrt{3}, y = -1$

$\frac{dx}{dt} = 2 \cos t, \frac{dy}{dt} = 4 \sin t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{4 \sin t}{2 \cos t} = 2 \tan t$

grad =  $2\sqrt{3}$

$\therefore y + 1 = 2\sqrt{3}(x - \sqrt{3})$

$y = 2\sqrt{3}x - 7$

d  $t = 3 \therefore x = 0, y = 4$

$\frac{dx}{dt} = -\frac{1}{4-t} = \frac{1}{t-4}, \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t}{\frac{1}{t-4}} = 2t(t-4)$

grad = -6

$\therefore y - 4 = -6(x - 0)$

$y = 4 - 6x$

$$4 \quad \theta = \frac{\pi}{3} \quad \therefore x = 2, y = 2\sqrt{3}$$

$$\frac{dx}{d\theta} = \sec \theta \tan \theta, \quad \frac{dy}{d\theta} = 2 \sec^2 \theta$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{2 \sec^2 \theta}{\sec \theta \tan \theta} \\ &= \frac{2 \sec \theta}{\tan \theta} = 2 \operatorname{cosec} \theta \end{aligned}$$

$$\text{grad} = \frac{4}{\sqrt{3}} \quad \therefore \text{grad of normal} = -\frac{\sqrt{3}}{4}$$

$$\therefore y - 2\sqrt{3} = -\frac{\sqrt{3}}{4}(x - 2)$$

$$4y - 8\sqrt{3} = -\sqrt{3}x + 2\sqrt{3}$$

$$\sqrt{3}x + 4y = 10\sqrt{3}$$

$$6 \quad \text{a} \quad \frac{dx}{dt} = 2 \cos 2t$$

$$\frac{dy}{dt} = 2 \sin t \times \cos t = \sin 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\sin 2t}{2 \cos 2t} = \frac{1}{2} \tan 2t$$

$$\text{b} \quad t = \frac{\pi}{6} \quad \therefore x = \frac{\sqrt{3}}{2}, y = \frac{1}{4}$$

$$\text{grad} = \frac{\sqrt{3}}{2}$$

$$\therefore y - \frac{1}{4} = \frac{\sqrt{3}}{2}(x - \frac{\sqrt{3}}{2})$$

$$[\sqrt{3}x - 2y - 1 = 0]$$

$$5 \quad \text{a} \quad \frac{dx}{dt} = -t^{-2}, \quad \frac{dy}{dt} = -(t+2)^{-2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-(t+2)^{-2}}{-t^{-2}}$$

$$= \frac{t^2}{(t+2)^2} = \left(\frac{t}{t+2}\right)^2$$

$$\text{b} \quad t = 2 \quad \therefore x = \frac{1}{2}, y = \frac{1}{4}$$

$$\text{grad} = \frac{1}{4} \quad \therefore \text{grad of normal} = -4$$

$$\therefore y - \frac{1}{4} = -4(x - \frac{1}{2})$$

$$4y - 1 = -16x + 8$$

$$16x + 4y - 9 = 0$$

$$7 \quad \text{a} \quad \frac{dx}{d\theta} = -3 \sin \theta, \quad \frac{dy}{d\theta} = 4 \cos \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = -\frac{4 \cos \theta}{3 \sin \theta}$$

$$\text{at } (3 \cos \alpha, 4 \sin \alpha), \quad \theta = \alpha$$

$$\therefore \text{grad} = -\frac{4 \cos \alpha}{3 \sin \alpha}$$

$$\therefore y - 4 \sin \alpha = -\frac{4 \cos \alpha}{3 \sin \alpha}(x - 3 \cos \alpha)$$

$$3y \sin \alpha - 12 \sin^2 \alpha = -4x \cos \alpha + 12 \cos^2 \alpha$$

$$3y \sin \alpha + 4x \cos \alpha = 12(\cos^2 \alpha + \sin^2 \alpha)$$

$$3y \sin \alpha + 4x \cos \alpha = 12$$

$$\text{b} \quad \text{at } (-\frac{3}{2}, 2\sqrt{3}),$$

$$3 \cos \alpha = -\frac{3}{2} \Rightarrow \cos \alpha = -\frac{1}{2}$$

$$4 \sin \alpha = 2\sqrt{3} \Rightarrow \sin \alpha = \frac{\sqrt{3}}{2}$$

$$\therefore 3y \times \frac{\sqrt{3}}{2} + 4x \times (-\frac{1}{2}) = 12$$

$$4x - 3\sqrt{3}y + 24 = 0$$

$$8 \quad \text{a} \quad x = 0 \Rightarrow t^2 = 0 \quad \Rightarrow t = 0$$

$$y = 0 \Rightarrow t(t-2) = 0 \Rightarrow t = 0, 2$$

$$\therefore (0, 0), (4, 0)$$

$$\text{b} \quad \text{i} \quad \frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2t - 2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t-2}{2t} = 1 - t^{-1}$$

$$t = x^{\frac{1}{2}} \quad \therefore \frac{dy}{dx} = 1 - x^{-\frac{1}{2}}$$

$$\text{ii} \quad y = x^{\frac{1}{2}}(x^{\frac{1}{2}} - 2) = x - 2x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 1 - x^{-\frac{1}{2}}$$

$$9 \quad \text{a} \quad \frac{dx}{d\theta} = 2 \sin \theta, \quad \frac{dy}{d\theta} = 3 \cos \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{3 \cos \theta}{2 \sin \theta} \quad \text{or} \quad \frac{3}{2} \cot \theta$$

$$\text{b} \quad \text{i} \quad \frac{3 \cos \theta}{2 \sin \theta} = 0 \quad \therefore \cos \theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \therefore (1, 3), (1, -3)$$

$$\text{ii} \quad \frac{3 \cos \theta}{2 \sin \theta} \rightarrow \infty \quad \therefore \sin \theta = 0$$

$$\theta = 0, \pi \quad \therefore (-1, 0), (3, 0)$$

$$10 \quad \mathbf{a} \quad x = 0 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0$$

$$y = 0 \Rightarrow \sin 2\theta = 0 \Rightarrow \theta = 0, \frac{\pi}{2}$$

$$\therefore (0, 0), (1, 0)$$

$$\mathbf{b} \quad \frac{dx}{d\theta} = \cos \theta, \quad \frac{dy}{d\theta} = 2 \cos 2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{2 \cos 2\theta}{\cos \theta}$$

$$\frac{2 \cos 2\theta}{\cos \theta} = 0 \quad \therefore \cos 2\theta = 0$$

$$\theta = \frac{\pi}{4} \quad \therefore y = 1$$

$$\mathbf{c} \quad y = \sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

$$0 \leq \theta \leq \frac{\pi}{2} \quad \therefore \cos \theta = \sqrt{1 - \sin^2 \theta}$$

$$\therefore y = 2x\sqrt{1-x^2}$$

$$12 \quad \mathbf{a} \quad t = 3, \quad x = \frac{10}{3}, \quad y = \frac{8}{3}$$

$$\frac{dx}{dt} = 1 - t^{-2}, \quad \frac{dy}{dt} = 1 + t^{-2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1+t^{-2}}{1-t^{-2}} = \frac{t^2+1}{t^2-1}$$

$$\text{grad} = \frac{5}{4}$$

$$\therefore y - \frac{8}{3} = \frac{5}{4} \left( x - \frac{10}{3} \right)$$

$$[ 5x - 4y - 6 = 0 ]$$

**b** sub. parametric eqns into tangent

$$5\left(t + \frac{1}{t}\right) - 4\left(t - \frac{1}{t}\right) - 6 = 0$$

$$5(t^2 + 1) - 4(t^2 - 1) - 6t = 0$$

$$t^2 - 6t + 9 = 0$$

$$(t - 3)^2 = 0$$

$\therefore t = 3$  (at  $P$ ), no other solutions

$\therefore$  does not intersect again

$$\mathbf{c} \quad x^2 = \left(t + \frac{1}{t}\right)^2 = t^2 + 2 + \frac{1}{t^2} \quad (1)$$

$$y^2 = \left(t - \frac{1}{t}\right)^2 = t^2 - 2 + \frac{1}{t^2} \quad (2)$$

$$(1) - (2) \Rightarrow x^2 - y^2 = 4 \quad [k = 4]$$

$$11 \quad \mathbf{a} \quad t = \frac{\pi}{4}, \quad x = \frac{1}{2}, \quad y = 1$$

$$\frac{dx}{dt} = 2 \sin t \cos t, \quad \frac{dy}{dt} = \sec^2 t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\sec^2 t}{2 \sin t \cos t} = \frac{1}{2} \sec^3 t \operatorname{cosec} t$$

$$\text{grad} = \frac{1}{2} \times (\sqrt{2})^3 \times \sqrt{2} = 2$$

$$\therefore y - 1 = 2\left(x - \frac{1}{2}\right)$$

$$y = 2x$$

when  $x = 0, y = 0$

$\therefore$  passes through origin

$$\mathbf{b} \quad y^2 = \tan^2 t = \frac{\sin^2 t}{\cos^2 t} = \frac{\sin^2 t}{1 - \sin^2 t}$$

$$\therefore y^2 = \frac{x}{1-x}$$